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ROTATIONS IN SPACE OF EVEN DIMENSIONS.

By H. B. Phillips and C. L. E. Moore.

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Introduction. Motions in three-space have been studied from three points of view: (a) the geometric in which the reasoning is primarily synthetic; (b) the algebraic expressed in terms of quaternions or Gibbs' dyadics; (c) that of groups of motions which was started by Jordan [1] (1884) and was continued by Lie and others. The literature of motions in three-space is quite extensive. The study of motions in n-space has not been nearly so extensive but has followed the general lines of development of motions in three-space. One of the first papers on the subject was by Cole [2] (1889). He discussed rotations in four-space by means of a sort of generalized Plücker coördinates. The principal result of the paper is that a rotation in four-space leaves two and only two completely perpendicular planes invariant. P. H. Schoute [3] (1891) by what might be called geometrical argument discussed the general motion in n-space. The algebraic method has not been used much so far as the authors of the present paper know. Clifford [4] and Lipschitz [5] generalized quaternions and thus established a transformation leaving the sum of squares invariant. The papers on this subject have dealt mostly with the algebra and very little with the geometry of motions.

The Gibbs' dyadic [6] is readily extended to n-space but its use for studying motions has not been very extensive. C. L. E. Moore [7] (1918) discussed rotations in hyperspace by treating first the resolu-

1 Annali di mathematica, 1884.

2 On rotations in space of four dimensions. American Journal, 12, 1889, page 191.

3 Le deplacement le plus général dans l'espace à n dimensions. Annales de l'Ecole Polytechnique de Delft, 7, 1891.

4 Applications of Grassmann's Extensive Algebra, American Journal 1, 350, 1878. Papers, London, 1882.

5 Untersuchungen ueber Summen von Quadraten. Bonn, Max Cohen & Sohn, 1886.

6 E. B. Wilson. On the theory of double products and strains in hyperspace. Trans. Conn. Acad. Arts Sci. 14, 1–57, 1908.

7 Rotations in Hyperspace. These Proceedings. 53, 1918.

tion of a complex two-vector into the sum of completely perpendicular simple two-vectors or planes. Gibbs' vector analysis and dyadics form a foundation of this work. E. B. Wilson [8] (1919) published a note outlining a discussion of rotations founded on the properties of the characteristic equation.

The group side of the subject has received more attention than the other two. Bemporad [9] (1904) returning to the original method of Jordan applied the general theory of Lie to determine the different types of groups of motions in three and four dimensions. E. E. Levi [10] (1905) considered the $\frac{1}{2}\ n(n+1)$ parametered group of motions in n-space. He treated the two and three parametered subgroups of the general group. The paper is intimately connected with a paper by Bianchi on generalized motion. C. L. E. Moore [11] (1918) discussed motions in n-space from the point of view of the infinitesimal transformation.

In this paper we use the Gibbs dyadic to discuss rotations in a space of $2n$ dimensions, particularly in four dimensions. It is shown first that any such rotation is the product of rotations in n fixed planes. The dyadic representing the general rotation is the sum of dyadics representing the rotations in these fixed planes. If we associate with each plane the two-vector in that plane equal in magnitude to the angle of rotation, the sum of such two-vectors for all the fixed planes is a complex two-vector M which plays a role analogous to that of the one-vector, equal to the angle of rotation, along the axis of a three dimensional rotation. A rotation in $2n$ dimensions may thus be described as a rotation through a two-vector angle M. Just as a rotation in the plane can be expressed in the form

$$\Xi' = e^{i\theta\cdot}\Xi$$

where Ξ and Ξ' are corresponding complex variables, so the rotation in $2n$ dimensions can be expressed in the form

$$r' = e^{I\cdot M}r$$

where r' and r are corresponding one-vectors. Furthermore, just as

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8 Note on Rotations in Hyperspace. Journal, Washington Acad. Sci. 9, 1919.

9 Sui gruppe dei movimenti. Annali della r. scuola normale sup. di Pisa, 8, 1904.

10 Sui gruppi di movimenti. Atti dei Lincei, series 5, 14, part 1, 1905.

11 Motions in Hyperspace. Annals of Math. second series, 19, 1918.

$i\theta$ represents a rotation through 90° and stretching of modulus θ, so $I \cdot M$ represents a rotation through 90° in each of the fixed planes and stretching in each of modulus equal to the angle of rotation.

In the study of commutative rotations it is found useful to define a new product of two-vectors which we call the star product. The star product of M_1 and M_2 is defined as the vector of the dyadic

$$(I \cdot M_1) \cdot (I \cdot M_2).$$

The cross product of two one-vectors in three dimensions is again a one-vector. No such product of one-vectors in higher space can, however, be defined. The star product combines two-vectors in three dimensions in the same way that the cross product combines one-vectors. It is however definable in space of any number of dimensions, the product being always a two-vector. It may therefore be considered a generalization in higher space of the Gibbs cross product. In four dimensions, if M_1 and M_2 are unit simple two-vectors and π_1 and π_2 perpendicular to and cutting both the star product is

$$M_1 * M_2 = \cos\theta_2 \, \pi_1 + \cos\theta_1 \, \pi_2,$$

θ_1 and θ_2 being the angles intercepted in π_1 and π_2 between M_1 and M_2.

If the rotations in the fixed planes are all through the same angle θ there is a fixed plane through each vector of space and all vectors are rotated through the same angle. These equiangular rotations have characteristic equations of order two. They are analogous to the rotations produced by quaternion multiplication [12] and are of two types corresponding to quaternions referred to right and left hand axes. In four dimensions equiangular rotations of opposite types are commutative and every rotation is the product of equiangular rotations of opposite types.

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12 H. B. Phillips, Application of quaternions to four dimensions. Johns Hopkins University Circular. January 1905, page 8.

In four dimensions a quaternion may be considered as a vector referred to system of four mutually perpendicular axes, one being the axis of reals, the other three the axes on which i, j, k lie. The product of two such vectors α and β is a generalization of the product of two complex numbers, the result being a vector which makes with each of the given vectors the same angle that the other makes with the axis of reals. Two vectors are thus determined one being $\alpha\beta$ the other $\beta\alpha$.

I. Rotations in $2n$ Dimensions.

1. **Characteristic equation.** In a space of $2n$ dimensions a rotation about the origin, considered as an operation on a one-vector, is represented by a matrix or dyadic ϕ of order $2n$. If r and r' are corresponding vectors the transformation is represented by

$$r' = \phi \cdot r$$

If this is a rotation r and r' will have the same length and therefore

$$r' \cdot r' = (\phi \cdot r) \cdot (\varphi \cdot r) = r \cdot \phi_c \, \phi \cdot r = r \cdot r.$$

Hence

$$\phi_c \, \phi = I. \tag{1}$$

(Here ϕ_c is used to represent the conjugate of ϕ, that is $\phi \cdot r = r \cdot \phi_c$ and I represents the identical transformation).

The dyadic ϕ satisfies a characteristic or Hamilton-Cayley equation

$$\phi^{2n} + a_1 \phi^{2n-1} + a_2 \phi^{2n-2} + \ldots . + a_{2n-1} \phi + a_{2n} = 0. \tag{2}$$

The conjugate of (2) is

$$\phi_c{}^{2n} + a_1 \phi_c{}^{2n-1} + a_2 \phi_c{}^{2n-2} + \ldots . + a_{2n-1} \phi_c + a_{2n} = 0.$$

Multiplying by ϕ^{2n} and reducing by use of (1)

$$1 + a_1 \phi + a_2 \phi^2 + \ldots . + a_{2n-1} \phi^{2n-1} + a_{2n} \phi^{2n} = 0. \tag{3}$$

Since this must be identical with (2)

$$a_{2n} = \pm 1. \tag{4}$$

It is well known that

$$(-1)^{2n} a_{2n} = a_{2n}$$

is the determinant of the transformation. If this is positive the rotation is called a proper rotation, if negative an improper one. In case of a proper motion the characteristic equation is then reciprocal. If λ is a root $\frac{1}{\lambda}$ is also a root. In case of an improper motion if λ is a root $-\frac{1}{\lambda}$ is also.

2. **Invariant planes.** If (2) has a real root λ, there must be a real vector r such that

$$r' = \phi \cdot r = \lambda r_1$$

Since r' and r have the same length

$$\lambda = \pm 1.$$

In case of a proper motion, there is a second real root

$$\frac{1}{\lambda} = \pm 1.$$

In case of an improper motion the second root is

$$-\frac{1}{\lambda} = \mp 1.$$

In either case there are two real roots (equal for a proper motion). There is then an invariant plane. That there is an invariant plane also follows from the fact that the line along which a lies is left invariant point for point and consequently the space of $2n - 1$ dimensions perpendicular to this is left invariant. In this space the characteristic equation is of odd order and so has a real root. Two vectors are thus left invariant and so their plane is invariant. In this plane each point is left fixed.

If (2) has no real root it must have two conjugate imaginary roots. Two conjugate imaginary vectors are therefore left invariant. The real plane containing these two vectors is then left invariant. Hence in any case a rotation in $2n$ dimensions has an invariant real plane. This plane is left invariant as a plane but the individual points are not left invariant. Since the $2n - 2$ dimensional space perpendicular to this invariant plane is also left invariant, by an application of the same argument there is an invariant plane in this perpendicular space. By continuing we see that *the rotation must leave n completely perpendicular real planes invariant.*[13]

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[13] It follows from this fact that positive rotations can be defined. For take two perpendicular unit vectors in each invariant plane arranged so that the rotation in each plane is from the first to the second. Then a positive rotation can be defined as one such that the outer product of these unit vectors (keeping the order indicated in each plane) is positive.

If the characteristic equation has a pair of roots equal to $\lambda + i\mu$ it will have a second pair equal to $\lambda - i\mu$. There will then be a pencil of imaginary lines corresponding to $\lambda + i\mu$ and another pencil corresponding to $\lambda - i\mu$. The imaginary planes in which these pencils lie are left invariant. Any plane through the origin cutting each of these invariant planes in a line will contain two invariant lines and so will be invariant. In particular the ∞^2 real planes cutting the planes of the invariant pencils in lines will be invariant. Thus repeated imaginary roots characterize the motions which have ∞^2 real invariant planes discussed by Moore.

3. **Canonical form of the dyadic.** Let k_1 and k_2 be perpendicular unit vectors. The dyadic

$$\Psi = (k_1 k_1 + k_2 k_2) \cos q + (k_2 k_1 - k_1 k_2) \sin q. \tag{5}$$

represents a rotation of the plane k_{12} though the angle q measured from k_1 toward k_2. That is if the vector r is rotated into r'

$$r' = \Psi \cdot r.$$

This is the most general form of a proper rotation of the plane into itself. Any improper rotation can be expressed in the form

$$x = (k_1 k_1 - k_2 k_2) \cos q + (k_2 k_1 + k_1 k_2) \sin q. \tag{6}$$

By properly choosing k_1 and k_2 this can always be reduced to a simpler form. Let

$$k_1' = k_1 \cos \tfrac{1}{2} q + k_2 \sin \tfrac{1}{2} q,$$
$$k_2' = k_1 \sin \tfrac{1}{2} q - k_2 \cos \tfrac{1}{2} q.$$

By direct expansion it is seen that

$$k_1' k_1' - k_2' k_2' = (k_1 k_1 - k_2 k_2) \cos q + (k_2 k_1 + k_1 k_2) \sin q.$$

If then we use k_1', k_2', instead of k_1, k_2 any improper rotation of the plane into itself can be expressed in the form

$$\chi = k_1 k_1 - k_2 k_2. \tag{7}$$

Let

$$k_{ij},\ i = 1, 3, \ldots 2n-1,\ j = i + 1, \tag{8}$$

be a set of n completely perpendicular planes and let k_i, k_j be perpendicular unit vectors in the plane k_{ij}. The dyadic

$$\phi = \Sigma (k_i k_i \pm k_j k_j) \cos q_{ij} + (k_j k_i \mp k_i k_j) \sin q_{ij}, \tag{9}$$

the summation being for all values of i and j in (8), represents a motion in which each plane k_{ij} is rotated through an angle q_{ij}. If the rotation in k_{ij} is proper the upper signs in formula (9) are used and if improper the lower signs are used. Any rotation Ψ in $2n$ dimensions can be expressed in this form. To do so let k_{ij} be the n invariant planes of Ψ and q_{ij} the angles of rotation in those planes. Then $\phi^{-1}\Psi$ is a dyadic leaving all the vectors k_i invariant. Hence

$$\phi^{-1}\Psi = I$$

and so $\phi = \Psi$.

If the rotations in two of the planes k_{ij} are improper, these can be replaced by two new planes in which the rotations are proper. Thus suppose k_{12} and k_{34} are the planes in question. By proper choice of k_1, k_2, k_3, k_4 the part of (9) belonging to these planes can be reduced, by (7), to the form

$$k_1 k_1 - k_2 k_2 + k_3 k_3 - k_4 k_4.$$

This is equivalent to

$$k_1 k_1 + k_3 k_3 - (k_2 k_2 + k_4 k_4)$$

which represents a proper motion leaving all the vectors in the plane k_{13} invariant and rotating the vectors in the plane k_{24} through 180°. The corresponding rotations will be proper. By a continuation of the process we can ultimately reduce (9) to a form in which not more than one of the rotations in the invariant planes will be improper. If the motion in only one of the planes is improper the $2n$ dimensional rotation represented by Ψ is improper. Hence *any proper rotation can be represented by a dyadic of the form*

$$\phi = \Sigma\,(k_i k_i + k_j k_j)\cos q_{ij} + (k_j k_i - k_i k_j)\sin q_{ij}. \qquad (10)$$

We shall now obtain the characteristic equation [14] satisfied by ϕ. In case of a rotation Ψ in a plane k_{12} we have

$$\Psi = (k_1 k_1 + k_2 k_2)\cos q_{12} + (k_2 k_1 - k_1 k_2)\sin q_{12},$$
$$\Psi^2 = (k_1 k_1 + k_2 k_2)(\cos^2 q_{12} - \sin^2 q_{12}) + (k_2 k_1 - k_1 k_2)\,2\sin q_{12}\cos q_{12},$$
$$I = k_1 k_1 + k_2 k_2.$$

Hence

$$\Psi^2 - 2\Psi\cos q_{12} + I = 0.$$

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14 See E. B. Wilson, note 8.

An equation satisfied by ϕ is then

$$\pi\,(\phi^2 - 2\,\phi\cos q_{ij} + I) = 0 \tag{11}$$

π indicating the product of all the factors obtained by letting i, j have the values in (8). For if x is any vector in the plane k_{ij}, we have

$$(\phi^2 - 2\,\phi\cos q_{12} + I)\cdot x = 0$$

and so

$$\pi\,(\phi^2 - 2\,\phi\cos q_{12} + I)\cdot x = 0 \tag{12}$$

Hence the left side of (11) is such that its product with any vector in any one of the fixed planes of ϕ is zero. Any vector in space can be represented as a linear function of vectors in these fixed planes. Consequently (12) is satisfied by any vector in space which proves relation (11).

If some of the angles q_{ij} are equal, (10) can be replaced by the equation of lower degree obtained by using only one factor for each distinct value of q_{ij}. For the above argument requires only one factor of each type. If one of the angles is zero the resulting factor

$$\phi^2 - 2\,\phi + I = (\phi - I)^2$$

can be replaced by $\phi - I$. For any vector x in the corresponding plane is left invariant and so

$$(\phi - I)\cdot x = x - x = 0.$$

If one of the roots is π, the factor

$$\phi^2 + 2\,\phi + I = (\phi + I)^2$$

can be replaced by $\phi + I$.

The resulting equation containing only one quadratic factor for each distinct value of q_{ij}, not more than one factor $\phi - I$ and not more than one factor $\phi + I$ is the equation of lowest degree satisfied by ϕ. For the roots of ϕ are

$$\cos q_{ij} + \sqrt{\cos^2 q_{ij} - 1} = e^{q_{ij}\sqrt{-1}}$$

and any equation satisfied by ϕ must be satisfied by all these roots.

4. **Exponential form of dyadic.** If Ψ is any dyadic, we shall write

$$e^{\Psi} = I + \Psi + \frac{\Psi^2}{\underline{2}} + \ldots . + \frac{\Psi^n}{\underline{n}} + \ldots \tag{13}$$

where I is the identical dyadic. The sum of n terms in (13) is a dyadic. That this sum approaches a definite limit, when n increases indefinitely has been shown by various writers.[15] By direct expansion it is seen that

$$e^{\Sigma q_{ij}\,(k_i k_j - k_j k_i)} = \Sigma\,(k_i k_i + k_j k_j)\cos q_{ij} + (k_j k_j - k_i k_j)\sin q_{ij}.$$

Hence the dyadic representing any proper rotation can be written in the form

$$\phi = e^{\Sigma q_{ij}\,(k_j k_i - k_i k_j)} \tag{14}$$

According to the inner multiplication of Lewis

$$\begin{aligned} k_i \cdot k_{ij} &= -k_j, \\ k_j \cdot k_{ij} &= \quad k_i, \\ k_l \cdot k_{ij} &= \quad 0, \quad l \neq i, j. \end{aligned}$$

The identical dyadic is

$$I = k_1 k_1 + k_2 k_2 + \ldots . k_{2n} k_{2n}.$$

Hence

$$I \cdot k_{ij} = k_j k_i - k_i k_j. \tag{15}$$

The product being obtained by multiplying the consequents of I with k_{ij}. If then we write

$$M = \Sigma q_{ij} k_{ij}, \tag{16}$$

equation (14) takes the form

$$\phi = e^{I \cdot M}. \tag{17}$$

This is a form in which the dyadic representing any proper rotation in $2n$ — dimensional space can be written.

In a space of $2n + 1$ dimensions the dyadic ϕ representing a given rotation has a characteristic equation of the form

$$\phi^{2n+1} + a_1 \phi^{2n} + \ldots . a_{2n}\,\phi + a_{2n+1} = 0.$$

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15 Cf. H. B. Phillips. Functions of Matrices, Amer. J. of Math. Vol. XLI.

In case of a proper rotation

$$a_{2n+1} = -1.$$

Since the equation is reciprocal, it has a root $+1$. Hence there is a vector i such that

$$\phi \cdot i = i.$$

The space perpendicular to i will be left invariant and in that space ϕ will determine a rotation with dyadic Ψ. Now

$$\phi = i\,i + \Psi.$$

For $i\,i + \Psi$ converts i into i and transforms the perpendicular space in the same way as Ψ. Also

$$\Psi = e^{I \cdot M}$$

where

$$I = k_1\,k_1 + k_2\,k_2 + \ldots. + k_{2n}\,k_{2n}$$

is the identical transformation in the space perpendicular to i. If however we let

$$I = k_1\,k_1 + k_2\,k_2 + \ldots. + k_{2n}\,k_{2n} + i\,i.$$

be the identical dyadic in the complete space of $2n+1$ dimensions, it is easily seen that

$$e^{I \cdot M} = i\,i + \Psi = \phi.$$

For M is a linear combination of the invariant planes which lie in the $2n$ − dimensional space perpendicular to i and $I \cdot M$ will be the same whether I is the $2n$ or the $2n+1$ − dimensional form of the identical dyadic. Then

$$e^{I \cdot M} = i\,i + I + I \cdot M + \frac{(I \cdot M)^2}{2} + \ldots.$$

where I on the right signifies the $2n$ − dimensional form. The above formula then follows at once. Hence, *whether the space rotated is of*

dimensions $2n$ *or* $2n+1$, *any proper rotation leaving the origin fixed can be determined by a dyadic of the form* (17).

Conversely any dyadic of the form

$$\phi = e^{I \cdot M}$$

represents a rotation. To show this, it is sufficient to express M as a linear function of completely perpendicular planes k_{ij}. Then

$$e^{I \cdot M} = e^{\Sigma q_{ij} (k_j k_i - k_i k_j)}$$

which is equivalent to (9) with all upper signs used.

The dyadic

$$I \cdot k_{ij} = k_j k_i - k_i k_j$$

determines a rotation through 90° in the plane k_{ij}. This is analogous to $i = \sqrt{-1}$ which produces a rotation of 90° in the complex domain. Similarly

$$\phi = e^{q I \cdot k_{ij}}$$

represents a rotation through the angle q in the plane k_{ij} just as $e^{i\varphi}$ produces a rotation through the angle φ in the complex domain. We can then consider $q\, k_{ij}$ as the vector angle of rotation. For this reason M might be called the *vector angle of rotation* of the dyadic

$$\phi = e^{I \cdot M}.$$

5. **Uniqueness of exponent.** The angles q_{ij} may be of any size. However, *if they are restricted to the region*

$$-\pi < q_{ij} < \pi$$

we shall show that *for each rotation there is a unique value of* M, that is, if

$$\phi = e^{I \cdot M_1} = e^{I \cdot M_2} \tag{18}$$

then

$$M_1 = M_2.$$

To show this we first consider the dyadic

$$I \cdot M = \Sigma q_{ij} (k_j k_i - k_i k_j).$$

Direct multiplication gives

$$(I \cdot M)^2 = \Sigma\, q_{ij}^2\, (-\, k_i\, k_i - k_j\, k_j).$$

Also

$$I = \Sigma\, (k_i\, k_i + k_j\, k_j).$$

The characteristic equation of $I \cdot M$ is then

$$\pi[(I \cdot M)^2 + q_{ij}^2\, I] = 0 \tag{19}$$

π representing the product for all values of q_{ij}. To obtain the equation of lowest degree satisfied by $I \cdot M$ we need only use one factor for each distinct value of q_{ij} and one factor $I \cdot M$ for each angle $q_{ij} = 0$. This is shown as in the case of the equation of lowest degree satisfied by ϕ by taking the product with any vector in an invariant plane. Hence the equation of lowest degree is of the same order as that of ϕ. Furthermore the values of q_{ij} (between $-\pi$ and $+\pi$) are determined by the roots

$$e^{q_{ij}\sqrt{-1}}$$

of ϕ. Consequently if (18) is satisfied, M_1 and M_2, satisfy the same equation of lowest degree (19) and the degree m of that equation is the same as that of ϕ. By using (19), terms of degree higher than $m - 1$ in

$$\phi = I + I \cdot M_1 + \frac{(I \cdot M_1)^2}{4} + \frac{(I \cdot M_1)^3}{\underline{|3}} + \ldots.$$

$$= I + I \cdot M_2 + \frac{(I \cdot M_2)^2}{\underline{|2}} + \frac{(I \cdot M_2)^3}{\underline{|3}} + \ldots.$$

can be replaced. Let the result be

$$\begin{aligned} \phi &= a_{11}\, I + a_{12}\, (I \cdot M_1) + a_{13}(I \cdot M)^2 + \ldots. + a_{1m}(I \cdot M_1)^{m-1} \\ &= a_{11}\, I + a_{12}\, (I \cdot M_2) + a_{13}(I \cdot M)^2 + \ldots. + a_{1m}(I \cdot M_2)^{m-1}. \end{aligned} \tag{20}$$

The coefficients a_{11}, a_{12}, etc., are the same in both cases because we reduce the same analytical expression by the use of the same identity

(19). Squaring both sides of (20) and using (19) as before

$$\begin{aligned}\phi^2 &= a_{21} I + a_{22}(I\cdot M_1) + \ldots + a_{2m}(I\cdot M_1)^{m-1} \\ &= a_{21} I + a_{22}(I\cdot M_2) + \ldots + a_{2m}(I\cdot M_2)^{m-1}.\end{aligned} \tag{21}$$

Continuing this process we have finally

$$\begin{aligned}\phi^{m-1} &= a_{m-11} I + a_{m-12}(I\cdot M_1) + \ldots + a_{m-1m}(I\cdot M_1)^{m-1} \\ &= a_{m-11} I + a_{m-12}(I\cdot M_2) + \ldots + a_{m-1m}(I\cdot M_2)^{m-1}.\end{aligned} \tag{22}$$

Since $\phi, \phi^2, \phi^3 \ldots . \phi^{m-1}, I$ are linearly independent the determinant

$$\begin{vmatrix} a_{12} & a_{13} \ldots\ldots a_{1m} \\ a_{22} & a_{23} \ldots\ldots a_{2m} \\ \ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots \\ a_{m-12} & a_{m-13} \ldots . a_{m-1m} \end{vmatrix} \neq 0. \tag{23}$$

From (20), (21), (22) we get

$$\begin{aligned} a_{12}[I\cdot M_1 - I\cdot M_2] + \ldots . + a_{1m}[(I\cdot M_1)^{m-1} - (I\cdot M_2)^{m-1}] &= 0, \\ a_{22}[I\cdot M_1 - I\cdot M_2] + \ldots . \quad a_{2m}[(I\cdot M_1)^{m-1} - (I\cdot M_2)^{m-1}] &= 0, \\ \ldots\ldots\ldots\ldots\

in the characteristic equation of ϕ. For the discussion of this case we may assume that there are no angles $q_{ij} = 0$. For such an angle corresponds to an absolutely fixed plane. If there are several such angles, the corresponding planes determine an absolutely fixed space of ϕ. Neither M_1 nor M_2 will have any planes lying in or cutting this space. The discussion is then identical with that of motions in the lower space perpendicular to this fixed space.

If there is only one angle equal to π, we have

$$M_1 = \pi\alpha_1 + q_2\alpha_2 + \ldots . q_n\alpha_n$$

where $\alpha_1, \alpha_2, \ldots \alpha_n$ are a set of completely perpendicular planes. This can be written

$$M_1 = \pi(\alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_n) + (q_2 - \pi)\alpha_2 + \ldots . + (q_n - \pi)\alpha_n.$$

It is easily seen that

$$e^{I \cdot \pi(\alpha_1 + \alpha_2 \ldots + \alpha_n)} = -I$$

since it represents a rotation of 180° in each of the planes $\alpha_1, \alpha_2 \ldots \alpha_n$. Hence

$$e^{I \cdot M} = -e^{I \cdot [(q_2 - \pi)\alpha_2 + \ldots . + (q_n - \pi)\alpha_n]}.$$

Similarly, we can write

$$M_2 = \pi\beta_1 + q_2\beta_2 + \ldots . + q_n\beta_n$$

and

$$e^{I \cdot M_2} = -e^{I \cdot [(q_2 - \pi)\beta_2 + \ldots . + (q_n - \pi)\beta_n]}.$$

If now

$$e^{I \cdot M_1} = e^{I \cdot M_2}$$

we can apply the previous argument and so find

$$(q_2 - \pi)\alpha_2 + \ldots . + (q_n - \pi)\alpha_n = (q_2 - \pi)\beta_2 + \ldots . + (q_n - \pi)\beta_n,$$

whence

$$M_1 - \pi(\alpha_1 + \alpha_2 + \ldots + \alpha_n) = M_2 - \pi(\beta_1 + \beta_2 + \ldots . + \beta_n).$$

Instead of $\alpha_1+\alpha_2+\ldots.+\alpha_n$ we could have used the function

$$\alpha_1 \pm \alpha_2 \pm \ldots. \pm \alpha_n,$$

the positive and negative signs being choosen independently providing we used the same signs for corresponding values of β_i.
Hence

$$M_1 - \pi(\alpha_1 \pm \alpha_2 \pm \ldots. \pm \alpha_n) = M_2 - \pi(\beta_1 \pm \beta_2 \pm \ldots \pm \beta_n)$$

and so

$$\alpha_2 = \beta_2, \alpha_3 = \beta_3, \ldots \alpha_n = \beta_n.$$

Since α_1 is the plane perpendicular to $\alpha_2, \alpha_3 \ldots \alpha_n$ and forming with them a right hand system, and β_1 has the same relation to $\beta_2, \beta_3, \ldots \beta_n$, we have finally $\alpha_1 = \beta_1$ and so

$$M_1 = M_2.$$

If two angles are equal to π the theorem may not be true. Thus in four dimensions

$$e^{\pi I \cdot (\alpha_1 + \alpha_2)} = e^{\pi I \cdot (\beta_1 + \beta_2)} = -I,$$

whether $\alpha_1+\alpha_2 = \beta_1+\beta_2$ or not. In higher dimensions the part of M_1 and M_2 in the four dimensional space of two planes rotated through 180° would be subject to the same indetermination.

6. **Vector of the dyadic.** If M is expressed as a linear function

$$M = \Sigma\, q_{ij}\, k_{ij}$$

of completely perpendicular planes k_{ij}, it is clear from the expression of ϕ in the form (9) that these planes are all left invariant and that the motion in each of them is proper.

The planes k_{ij} are invariant in the sense that if k_i and k_j go by the rotation into k_i' and k_j', since these new vectors are perpendicular and lie in the plane k_{ij},

$$k_i' \times k_i' = k_i \times k_j$$

For the same reason M is invariant. Also any linear function of the

planes k_{ij} is invariant. A particular linear function of considerable interest is the vector of the dyadic.

$$\phi_x = \Sigma\, 2 \sin q_{ij}\, k_{ij} \tag{24}$$

obtained by replacing each dyad of ϕ by the vector product of its two factors. The rotation cannot be determined uniquely when the vector is known, however, for the angles q_{ij} might be replaced by their supplements without changing the vector of the dyadic.

7. **Commutative rotations and star product.** Consider two rotations

$$\phi_1 = e^{I \cdot M_1},\ \phi_2 = e^{I \cdot M_2}.$$

Since ϕ_1 and ϕ_2 are sums of powers of $I \cdot M_1$ and $I \cdot M_2$ they will be commutative if $I \cdot M_1$ and $I \cdot M_2$ are commutative, that is if

$$\Psi = (I \cdot M_1) \cdot (I \cdot M_2) - (I \cdot M_2 - I \cdot M_1) = 0. \tag{25}$$

Represent M_1 and M_2 in the symbolic form

$$M_1 = AB,\ M_2 = CD.$$

Then

$$I \cdot M_1 = BA - AB,\ I \cdot M_2 = DC - CD,$$

and

$$\begin{aligned} \Psi &= (A \cdot D)(BC - CB) + (B \cdot C)(AD - DA) - (A \cdot C)(BD - DB) - \\ &\qquad (B \cdot D)(AC - CA) \\ &= I \cdot [(A \cdot D)(C \times B) + (B \cdot C)(D \times A) - (A \cdot C)(D \times B) - \\ &\qquad (B \cdot D)(C \times A)] \\ &= -I \cdot [(I \cdot M_1) \cdot (I \cdot M_2)]_\times \end{aligned} \tag{26}$$

where $[\ ]_\times$ signifies the vector of the dyadic contained in the brackets. Let

$$\begin{aligned} M_1 * M_2 &= -[(I \cdot M_1) \cdot (I \cdot M_2)]_\times = [(I \cdot M_2) \cdot (I \cdot M_1)]_\times \\ &= (A \cdot C)(B \times D) + (B \cdot D)(A \times C) - (A \cdot D)(B \times C) - \\ &\qquad (B \cdot C)(A \times D). \end{aligned} \tag{27}$$

We will call this the *star product* of M_1 and M_2. It is evidently

distributive since the operations in terms of which it is expressed are distributive. Equation (26) shows that ϕ_1 *and* ϕ_2 *are commutative if*

$$M_1 * M_2 = 0.$$

Conversely, if ϕ_1 and ϕ_2 are commutative and none of the angles equal to π.

$$M_1 * M_2 = 0.$$

For we can write ϕ_1 in the form

$$\phi_1 = a_{11} I + a_{12} (I \cdot M_1) + \ldots . + a_{1 \cdot m} (I \cdot M_1)^{m-1}$$

as in (20). If then ϕ_2 is commutative with ϕ_1

$$\phi_2 \phi_1 - \phi_1 \phi_2 = a_{12}[\phi_2 (I \cdot M_1) - (I \cdot M_1)\phi_2] + \ldots . + a_{1m}[\phi_2 (I \cdot M_1)^{m-1} - (I \cdot M)^{m-1}\phi_2] = 0.$$

Since ϕ_2 will also be commutative with ϕ_1^2, from (21)

$$a_{22} [\phi_2 (I \cdot M_1) - (I \cdot M_1) \phi_2 + \ldots . + a_{2m}[\phi_2 (I \cdot M_1)^{m-1} \cdot - (I \cdot M_1)^{m-1}\phi_2] = 0.$$

Similarly for other powers of ϕ_1. Finally

$$a_{mm-1} [\phi_2 (I \cdot M_1) - (I \cdot M_1)\phi_2] + \ldots + a_{m-1m}[\phi_2 (I \cdot M_1)^{m-1} - (I \cdot M_1)^{m-1}\phi_2] = 0.$$

There are $m - 1$ equations linear and homogeneous in $m - 1$ unknowns $[\phi_2 (I \cdot M_1) - (I \cdot M_1) \phi_2]$, *etc.* Since the determinant (23) is not zero each of the unknowns must be zero. Hence

$$\phi_2 (I \cdot M_1) - (I \cdot M_1) \phi_2 = 0. \tag{28}$$

This shows that ϕ_2 is commutative with $I \cdot M_1$. Expanding ϕ_2 in the form (20) and repeating the argument, we then get,

$$(I \cdot M_1) (I \cdot M_2) - (I \cdot M_2) (I \cdot M_1) = 0,$$

that is $I \cdot M_1$ and $I \cdot M_2$ are commutative. Equation (25) is then satisfied and so

$$M_1 * M_2 = 0,$$

since this equation was shown to be a consequence of (25).

The star product is defined by (27) for the product of two planes or complex two vectors. We can obtain $M_2 * M_1$ by interchanging A with C and B with D. It follows from (27) that

$$M_2 * M_1 = - M_1 * M_2. \tag{29}$$

In particular if $M_1 = M_2 = M$, we get

$$M * M = 0. \tag{30}$$

Representing M_3 in the symbolic form EF, by direct expansion we get

$$M_1 \cdot (M_2 * M_3) = M_3 \cdot (M_2 * M_1). \tag{31}$$

8. **Star product of two planes.** Suppose the two planes M_1 and M_2 lie in a 3-space. They can be written

$$M_1 = A \times B, \quad M_2 = A \times C$$

where A is a unit vector and B and C are perpendicular to A. Then

$$M_1 * M_2 = B \times C.$$

This is a plane vector perpendicular to M_1 and M_2 and having a magnitude equal to the product of the magnitudes of M_1 and M_2 and the sine of the angle between them. In particular the product vanishes when and only when M_1 and M_2 lie in the same plane.

If M_1 and M_2 are not contained in three dimensions they always lie in a 4-space. In that space we can always find two planes $A \times C$ and $B \times D$ perpendicular to and cutting both. We can then choose the vectors A, B, C, D such that

$$M_1 = A \times B, \quad M_2 = C \times D.$$

Then since

$$A \cdot B = A \cdot D = C \cdot B = C \cdot D = 0,$$
$$M_1 * M_2 = (A \cdot C)\, B \times D + (B \cdot D)\, A \times C. \tag{32}$$

The product is a linear function of the two vectors perpendicular to both. In particular the product is zero only when

$$A \cdot C = B \cdot D = 0.$$

that is, when M_1 and M_2 are completely perpendicular. Thus *the star product of two plane vectors vanishes when and only when they lie in the same plane or are completely perpendicular.*

An interesting case occurs when

$$A \cdot C = \pm B \cdot D \neq 0.$$

For simplicity let A, B, C, D be unit vectors. Any unit vector in M_1 can be written

$$X = A \cos \alpha + B \sin \alpha$$

and similarly a vector in M_2 can be written

$$Y = C \cos \beta + D \sin \beta.$$

The angle θ between these vectors satisfies the relation

$$\begin{aligned} \cos \theta &= A \cdot C \cos \alpha \cos \beta + B \cdot D \sin \alpha \sin \beta \\ &= A \cdot C \, (\cos \alpha \cos \beta \pm \sin \alpha \sin \beta) \\ &= A \cdot C \cos (\alpha \pm \beta). \end{aligned}$$

The minimum value of θ is given by

$$\cos \theta = \pm A \cdot C.$$

This is the angle between X and M_2. Hence all vectors in M_1 make the same angle with M_2 and vice versa. Since $A \times C$ and $B \times D$ are completely perpendicular unit planes, we could choose the vectors A, B, C, D such that $A \times C$ is the complement of $B \times D$ in their four dimensional space. If then $A \cdot C = B \cdot D$, $M_1 * M_2$ is equal to its complement, that is, is self-complementary. If $A \cdot C = -B \cdot D$, $M_1 * M_2$ is the negative of its complement that is, it is anti-self-complementary. For all vectors in either plane to make the same angle with the other plane it is then necessary and sufficient that $M_1 * M_2$ be self-complementary or anti-self-complementary.

In four dimensions all vectors X through the origin making a given angle with M_2 belong to the quadric cone

$$* \ (M_2 \cdot X) \cdot (M_2 \cdot X) = 0.$$

This cone has two sets of generating planes. Any plane M of the one set is such that $M * M_2$ is self-complementary and any plane of the other set such that $M * M_2$ is anti-self-complementary.

II. Rotations in four dimensions.

9. **Identities.** In terms of unit two-vectors k_{ij} the laws of star multiplication are

$$\begin{aligned} k_{12} * k_{12} &= k_{12} * k_{34} = 0, \\ k_{12} * k_{13} &= k_{23}. \end{aligned} \tag{33}$$

The case of four dimensions is the one in which we shall be most interested. Let

$$M = \Sigma \lambda_{ij} k_{ij}$$

be any two-vector in four dimensions. Then

$$\begin{aligned} k_{12} * M &= \lambda_{13} k_{23} + \lambda_{14} k_{24} - \lambda_{23} k_{13} - \lambda_{24} k_{14}, \\ k_{34} * M &= -\lambda_{13} k_{41} + \lambda_4 k_{31} - \lambda_{23} k_{42} + \lambda_{24} k_{32}. \end{aligned} \tag{34}$$

Denoting the complement by the subscript c we then have

$$(k_{12} * M)_c = (k_{34} * \mathrm{M}) = (k_{12})_c * M.$$

Since any unit plane could be taken as k_{12}

$$(M_1 * M_2)_c = (M_1)_c * M_2 = M_1 * (M_2)_c. \tag{35}$$

The operations in (35) being distributive the equations are valid whether M_1 and M_2 are planes or complex two-vectors in four dimensions.

Since the complement of $(M_1 * M_2)_c$ is $M_1 * M_2$ we also have

$$M_1 * M_2 = (M_1)_c * (M_2)_c. \tag{36}$$

10. **Geometrical significance of $M_1 * M_2 = 0$.** Let M_1 and M_2 be complex two-vectors. They can be expressed in the form

$$M_1 = \lambda_1 \pi_1 + \mu_1 (\pi_1)_c$$
$$M_2 = \lambda_2 \pi_2 + \mu_2 (\pi_2)_c.$$

Expanding and using (35) and (36) we have

$$M_1 * M_2 = (\lambda_1 \lambda_2 + \mu_1 \mu_2)\ \pi_1 * \pi_2 + (\lambda_1 \mu_2 + \lambda_2 \mu_1)\ \pi_1 * (\pi_2)_c, \quad (37)$$

Now $\pi_1 * \pi_2$ is a linear function of two completely perpendicular planes perpendicular to and cutting both π_1 and π_2. Since

$$\pi_1 * (\pi_2)_c = (\pi_1 * \pi_2)_c$$

it is a linear function of the same planes. Therefore *if M_1 and M_2 are resolved in any way into linear functions of pairs of completely perpendicular planes $M_1 * M_2$ is a linear function of two completely perpendicular planes, perpendicular to and cutting the planes of each pair.*

Suppose now

$$M_1 * M_2 = 0.$$

The complement of this is

$$(M_1)_c * M_2 = 0.$$

Let

$$M_1 = \alpha k_{12} + \beta k_{34};\ M_2 = \Sigma \lambda_{ij} k_{ij}.$$

Then $(M_1)_c = \alpha k_{34} + \beta k_{12}$ and

$$0 = [\alpha M_1 - \beta (M_1)_c] * M_2 = (\alpha^2 - \beta^2)\, k_{12} * M_2.$$

If M_1 is not self-complementary or anti-self-complementary

$$\alpha^2 - \beta^2 \neq 0,\ k_{12} * M_2 = 0.$$

From (34) we then have

$$\lambda_{13} = \lambda_{14} = \lambda_{23} = \lambda_{24} = 0.$$

Hence

$$M_2 = \lambda_{12} k_{12} + \lambda_{34} k_{34}.$$

Consequently M_1 and M_2 are linear functions of the same pair of completely perpendicular planes k_{12} and k_{34}. We could obtain a similar conclusion if M_2 was not self-complementary or anti-self-complementary. If both belong to these special types, let

$$M_1 = \alpha\,(k_{12} + k_{34}).$$

From (34) we then have

$$M_1 * M_2 = \alpha\,[(\lambda_{13} - \lambda_{24})\,k_{23} + (\lambda_{14} + \lambda_{23})\,k_{24} + (\lambda_{23} + \lambda_{14})\,k_{31} + (\lambda_{24} - \lambda_{13})\,k_{41}.$$

If this is zero

$$\lambda_{13} = \lambda_{24},\ \lambda_{14} = -\lambda_{23}.$$

Hence M_2 is anti-self-complementary and so

$$M_2 = \lambda_{12}\,(k_{12} - k_{34}) + \lambda_{13}\,(k_{13} + k_{24}) + \lambda_{14}\,(k_{14} - k_{23}).$$

By direct multiplication we get

$$\begin{aligned} M_1 \cdot M_2 &= M_1 \times M_2 = M_1 * M_2 = 0, \\ M_1 \times M_1 &= M_1 \cdot M_1 = 2\,\alpha^2 > 0, \\ M_2 \times M_2 &= -M_2 \cdot M_2 = -2\,(\lambda_{12}^2 + \lambda_{13}^2 + \lambda_{14}^2) < 0. \end{aligned}$$

Hence

$$(M_1 + \lambda\,M_2) \times (M_1 + \lambda\,M_2) = M_1 \times M_1 + \lambda^2\,M_2 \times M_2 = 0$$

has two real roots λ. If then

$$\pi_1 = M_1 + \lambda\,M_2,\ \pi_2 = M_1 - \lambda\,M_2,$$

π_1 and π_2 are perpendicular planes and M_1 and M_2 linear functions of them. Hence in any case *if the star product of two two-vectors in four dimensions is zero, they are expressible as linear functions of the same pair of completely perpendicular planes.* Incidentally we have shown that the star product, cross product and dot product of a self-complementary and an anti-self-complementary two-vector are all zero.

Let π_1', π_2', π_3' be the complements of π_1, π_2, π_3. We shall now prove the identity

$$\pi_1 * (\pi_2 * \pi_3) = (\pi_1 \cdot \pi_3)\pi_2 - (\pi_1 \cdot \pi_2)\pi_3 + (\pi_1 \cdot \pi_3')\pi_2' - (\pi_1 \cdot \pi_2')\pi_3'. \quad (38)$$

Since both sides are distributive functions of π_1, π_2, π_3 it is sufficient to prove the identity when they are products of units. If $\pi_2 = \pi_3$ or $\pi_2 = \pi_3'$ both sides vanish. In any other case with a proper choice of axes we could write

$$\pi_2 = k_{12}, \quad \pi_3 = k_{13}, \quad \pi_1 = k_{ij}.$$

Since the subscripts 2 and 3 enter in a similar manner, we need consider only the four cases $\pi_1 = k_{12}$, k_{14}, k_{23} or k_{24}. In these cases the identity becomes

$$-k_{13} = -k_{13}, \quad 0 = 0, \quad 0 = 0, \quad k_{43} = -k_{34}$$

respectively.

By applying (38) to each term and adding, since $\pi_1 \cdot \pi_2' = \pi_1' \cdot \pi_2$ ecc., we get

$$\pi_1 * (\pi_2 * \pi_3) + \pi_2 * (\pi_3 * \pi_1) + \pi_3 * (\pi_1 * \pi_2) = 0. \tag{39}$$

Another identity is

$$\begin{aligned}(\pi_1 * \pi_2) * (\pi_3 * \pi_4) = {} & (\pi_1 \cdot \pi_3)\, \pi_2 * \pi_4 - (\pi_1 \cdot \pi_4)\, \pi_2 * \pi_3 \\ & + (\pi_2 \cdot \pi_4)\, \pi_1 * \pi_3 - (\pi_2 \cdot \pi_3)\, \pi_1 * \pi_4 \\ & + (\pi_1 \cdot \pi_3')\, \pi_2 * \pi_4' - (\pi_1 \cdot \pi_4')\, \pi_2 * \pi_3' \\ & + (\pi_2 \cdot \pi_4')\, \pi_1 * \pi_3' - (\pi_2 \cdot \pi_3')\, \pi_1 * \pi_4'.\end{aligned} \tag{40}$$

Both sides vanish when $\pi_1 = \pi_2$, $\pi_1 = \pi_2'$, $\pi_3 = \pi_4$ or $\pi_3 = \pi_4'$. Since there are only three units and their complements, we must have in any other case π_1 or π_2 equal to π_3, π_3', π_4 or π_4'. These cases are all handled in practically the same way. We consider then only the case $\pi_1 = \pi_3$. The identity then reduces to

$$(\pi_1 * \pi_2) * (\pi_1 * \pi_4) = (\pi_1 \cdot \pi_1)\, \pi_2 * \pi_4. \tag{41}$$

Both sides vanish when $\pi_2 = \pi_4$ or $\pi_2 = \pi_4'$. We could then take

$$\pi_1 = k_{12}, \quad \pi_2 = k_{13}, \quad \pi_4 = k_{14} \text{ or } k_{23}.$$

In these cases (41) becomes

$$k_{23} * k_{24} = k_{34}$$

and

$$k_{23} * k_{31} = k_{12}$$

respectively, both of which are correct. Formula (40) is then correct for all unit two-vectors and since it is distributive it is therefore true when π_1, π_2, π_3, π_4 are complex two vectors.

11. **Rotation of a vector.** Let

$$\phi = e^{I \cdot M} = e^{I \cdot (q_1 \pi_1 + q_2 \pi_2)}$$

where π_1 and π_2 are two completely perpendicular unit planes in terms of which M is linearly expressible. Suppose ϕ rotates a unit vector r through an angle θ into a unit vector

$$r' = \phi \cdot r.$$

Then

$$\cos\theta = r \cdot r' = r \cdot \phi \cdot r. \tag{42}$$

Let α be the angle between r and the invariant plane π_1. We can then write

$$r = r_1 + r_2$$

where r_1 and r_2 are the projections of r on the invariant planes π_1 and π_2. Since $\phi \cdot r_1$ and $\phi \cdot r_2$ are in the planes π_1, and π_2 respectively

$$r_1 \cdot \phi \cdot r_2 = 0,$$

and so

$$\begin{aligned} \cos\theta &= r \cdot \phi \cdot r = r_1 \cdot \phi \cdot r_1 + r_2 \cdot \phi \cdot r_2 \\ &= r^2 \cos^2\alpha \cos q_1 + r^2 \sin^2\alpha \cos q_2. \end{aligned}$$

Hence

$$\cos\theta = \cos^2\alpha \cos q_1 + \sin^2\alpha \cos q_2. \tag{43}$$

The angle θ then depends only on α, q_1 and q_2. Since π_1 is invariant r rotates into a vector r' making the same angle α with π_1. Consequently the motion rotates r' through the same angle θ that it rotates r through.

If q_1 and q_2 are not equal it is clear from (43) that the maximum and minimum valves of θ are determined by $\alpha = 0, \frac{\pi}{2}$, that is, when r lies

in π_1 or π_2. *Vectors in one of the invariant planes are rotated through the maximum angle and vectors in the other are rotated through the minimum angle and those in neither plane are rotated through an intermediate angle.* If $q_1 = \pm q_2$

$$\cos\theta = \cos q_1 = \cos q_2$$

and it would appear that all vectors are rotated through the same angle. In this case M is self-complementary or anti-self-complementary.

12. **Equiangular rotations.** If M is any two-vector and k_1 a unit one-vector, M can be expressed in terms of four perpendicular vectors including k_1. The terms of M containing k_1 will have for sum a plane since the sum of three plane vectors with a vector in common, is a plane vector. Let k_2 be perpendicular to k_1 in that plane and let k_3, k_4, form with k_1 and k_2 a system of orthogonal unit vectors. Then

$$M = \lambda_{12}\, k_{12} + \lambda_{23}\, k_{23} + \lambda_{24}\, k_{24} + \lambda_{34}\, k_{34}$$

and

$$M_c = \lambda_{12}\, k_{34} + \lambda_{23}\, k_{14} + \lambda_{24}\, k_{31} + \lambda_{34}\, k_{12}.$$

If $M = M_c$

$$\lambda_{12} = \lambda_{34}, \quad \lambda_{23} = \lambda_{24} = 0$$

and

$$M = \lambda_{12}\,(k_{12} + k_{34}).$$

If $M = -M_c$

$$\lambda_{12} = -\lambda_{34}, \quad \lambda_{23} = \lambda_{24} = 0$$

and

$$M = \lambda_{12}\,(k_{12} - k_{34}).$$

In either case M is a linear function of two completely perpendicular planes one of which may be taken through the arbitrary vector k_1. By properly choosing k_2 we can always make λ_{12} positive. Then λ_{12} will have the same value for all choices of k_1. For if k'_{12} and k'_{34} are two other completely perpendicular planes and

$$M = \lambda\,(k_{12} \pm k_{34}) = \mu\,(k'_{12} \pm k'_{34}),$$
$$M \cdot M = 2\lambda^2 = 2\mu^2.$$

If now

$$M = \pm M_c, \quad \phi = e^{I \cdot M} \tag{44}$$

we can write ϕ in the form

$$\phi = e^{I \cdot M} = e^{qI \cdot (k_{12} \pm k_{34})}$$

where k_1 is an arbitrarily chosen vector and q is a constant of the motion. The motion rotates k_1 through the angle q in the fixed plane k_{12}. Hence *every vector moves in a fixed plane and all are rotated through the same angle q*. For this reason we shall call the rotation equiangular when

$$M = \pm M_c. \tag{45}$$

Conversely, if all vectors are rotated through the same angle q, in particular the rotation in the fixed plane must be through the angle

$$q_1 = q_2 = q$$

and so (44) is satisfied.

In the case just discussed, if x is any vector, since x, ϕx, $\phi^2 x$ lie in a plane and the angle from x to ϕx is q and that from ϕx to $\phi^2 x$ is also q,

$$x + \phi^2 x = 2 \cos q \, \phi \, x.$$

Since this is true for all values of x

$$\phi^2 = 2 \phi \cos q - I. \tag{46}$$

This is the characteristic equation of ϕ. Conversely if any rotation has this for characteristic equation, x, ϕx, $\phi^2 x$, lie in the same plane. This plane will then be invariant and the rotation in the plane will be through the angle q. The rotation will therefore be equiangular.

These rotations resemble very much the rotations that result from quaternion multiplication. If the angle of rotation is q, by (44)

$$M \cdot M = 2 q^2. \tag{47}$$

In terms of a given set of orthogonal vectors k_1, k_2, k_3, k_4, any self-complementary two-vector of this kind be written,

$$M = q [\lambda (k_{12} + k_{34}) + \mu (k_{13} - k_{24}) + \nu (k_{14} + k_{23})]$$

where

$$\lambda^2 + \mu^2 + \nu^2 = 1. \tag{48}$$

Expanding and using (48) we get

$$\phi = e^{I \cdot M} = I \cos q + \lambda \sin q\, I \cdot (k_{12} + k_{34}) + \mu \sin q\, \mathrm{I} \cdot (k_{13} - k_{24}) + \nu \sin q\, I \cdot (k_{14} + k_{23}).$$

If we take

$$i = I \cdot (k_{12} + k_{34}),$$
$$j = I \cdot (k_{13} - k_{24}),$$
$$k = I \cdot (k_{14} + k_{23}),$$

then

$$\phi = I \cos q + \sin q\, (\lambda i + \mu j + \nu k).$$

Also by direct multiplication it is seen that

$$i^2 = j^2 = k^2 = -I,$$
$$ij = k, \quad jk = i, \quad ki = j,$$

as in quaternions. Two dyadics of this kind can then be multiplied like quaternions the principal difference being that 1 in quaternions is replaced by I. If M is anti-self-complementary we get similar results with

$$i = I \cdot (k_{12} - k_{34}),$$
$$j = I \cdot (k_{13} + k_{24}),$$
$$k = I \cdot (k_{23} - k_{14}).$$

Rotations with self-complementary M differ from those with anti-self-complementary M in the same way that quaternions referred to a right hand system of axes differ from quaternions referred to a left hand system.

13. **Invariant complexes.** If M is self-complementary the rotation

$$\phi = e^{I \cdot M}$$

leaves all anti-self-complementary two-vectors invariant. For if M' is such a two-vector M and M' can be expressed in the form

$$M = q\, (\pi_1 + \pi_2), \quad M' = \lambda\, (\pi_1 - \pi_2)$$

where π_1 and π_2 are completely perpendicular unit planes. Now π_1 and π_2 being fixed planes of ϕ, are left invariant. Hence ϕ leaves M' invariant. Similarly the rotation

$$\phi = e^{I \cdot M'}$$

leaves any self-complementary two-vector M invariant.

Let

$$\phi = e^{I \cdot M} = e^{I \cdot k_{12}(q_1 k_{12} + q_2 k_{34})}.$$

The rotation evidently leaves the self-complementary-complex

$$k_{12} + k_{34}$$

invariant. Any self-complementary complex can be written

$$\lambda (k_{12} + k_{34}) + \mu (k_{13} - k_{24}) + \nu (\lambda_{14} + k_{23}).$$

If ϕ leaves this invariant, it must leave

$$\mu (k_{13} - k_{24}) + \nu (k_{14} + k_{23})$$

invariant. Since k_1 transforms into

$$k_1 \cos q_1 + k_2 \sin q_1, \text{ etc.}$$

we must then have

$$\begin{aligned}
&\mu(k_{13} - k_{24}) + \nu(k_{14} + k_{23}) = \\
&= \mu [(k_1 \cos q_1 + k_2 \sin q_1) \times (k_3 \cos q_2 + k_4 \sin q_2) - (k_2 \cos q_1 \\
&\qquad - k_1 \sin q_1) \times (k_4 \cos q_2 - k_3 \sin q_2)] \\
&+ \nu [(k_1 \cos q_1 + k_2 \sin q_1) \times (k_4 \cos q_2 - k_3 \sin q_2) + (k_2 \cos q_1 \\
&\qquad - k_1 \sin q_1) \times (k_3 \cos q_2 + k_4 \sin q_2).
\end{aligned}$$

Equating coefficients of k_{13}, we get

$$\mu \cos (q_1 + q_2) - \nu \sin (q_1 + q_2) = \mu.$$

Similarly from the coefficients of k_{14} we get

$$\mu \sin (q_1 + q_2) + \nu \cos (q_1 + q_2) = \nu.$$

If these equations are satisfied simultaneously

$$\begin{vmatrix} \cos(q_1 + q_2) - 1, & -\sin(q_1 + q_2) \\ \sin(q_1 + q_2), & \cos(q_1 + q_2) - 1 \end{vmatrix} = 0,$$

that is,

$$\cos(q_1 + q_2) = 1.$$

Hence

$$q_1 = -q_2.$$

Consequently M is anti-self-complementary. If then ϕ leaves any self-complementary complex invariant other than $k_{12} + k_{34}$, M must be anti-self-complementary. In particular *if M is self-complementary, the rotation cannot leave any self complementary complex invariant except a multiple of M.* Similar theorems hold for anti-self-complementary rotations.

The self-complementary complexes are analogous to one system of the generators on a quadric surface, the anti-self-complementary complexes constituting the generators of the other system, each generator of one system cutting each generator of the other in the sense that the star product is zero. The equiangular rotations, are the rotations that leave this quadric invariant. Any rotation of the self-complementary type leave all generators of the opposite type invariant and one of the self-complementary type. Similarly for the anti-self-complementary rotations.

14. **Analogy to Gibbs' vector semi-tangent.**[16] Any self-complementary complex can be written

$$M = q(k_{12} + k_{34}).$$

Hence

$$M \cdot M = M \times M = 2q^2, \tag{49}$$

$$I \cdot M = q(k_2 k_1 - k_1 k_2 + k_4 k_3 - k_3 k_4), \tag{50}$$

$$(I \cdot M) \cdot (I \cdot M) = -q^2 I = -\tfrac{1}{2}(M \cdot M) I. \tag{51}$$

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If M_1 and M_2 are self-complementary $M_1 + M_2$ is also and so

$$I\cdot(M_1 + M_2)\cdot I\cdot(M_1 + M_2) = -\tfrac{1}{2}(M_1 + M_2)\cdot(M_1 + M_2)\, I.$$

Since

$$(I\cdot M_1)\cdot(I\cdot M_1) = -\tfrac{1}{2}(M_1\cdot M_1)\, I,$$
$$(I\cdot M_2)\cdot(I\cdot M_2) = -\tfrac{1}{2}(M_2\cdot M_2)\, I.$$

by subtraction we get

$$(I\cdot M_1)\cdot(I\cdot M_2) + (I\cdot M_2)\cdot(I\cdot M_1) = -(M_1\cdot M_2)\, I. \tag{52}$$

also

$$(I\cdot M_1)\cdot(I\cdot M_2) - (I\cdot M_2)\cdot(I\cdot M_1) = I\cdot(M_1 * M_2),$$

whence by addition we get

$$(I\cdot M_1)\cdot(I\cdot M_2) = \tfrac{1}{2}[I\cdot(M_1 * M_2 - (M_1\cdot M_2)\, I]. \tag{53}$$

Let

$$M_1\cdot M_1 = M_2\cdot M_2 = 2.$$

Then

$$\phi_1 = e^{q_1 I\cdot M_1} = I\cos q_1 + I\cdot M_1 \sin q_1,$$
$$\phi_2 = e^{q_2 I\cdot M_2} = I\cos q_2 + I\cdot M_2 \sin q_2.$$

Multiplying and using (53) we get

$$\phi_1\phi_2 = I[\cos q_1 \cos q_2 - \tfrac{1}{2}(M_1\cdot M_2)\sin q_1 \sin q_2] \tag{54}$$
$$+ I\cdot[M_1 \sin q_1 \cos q_2 + M_2 \cos q_1 \sin q_2 + \tfrac{1}{2} M_1 * M_2 \sin q_1 \sin q_2.$$

Let

$$Q_1 = M_1 \tan q_1, \qquad Q_2 = M_2 \tan q_2.$$

The corresponding quantity for $\phi_1\,\phi_2$ is

$$Q = \frac{M_1 \sin q_1 \cos q_2 + M_2 \cos q_1 \sin q_2 + \tfrac{1}{2}(M_1 * M_2)\sin q_1 \sin q_2}{\cos q_1 \cos q_2 - \tfrac{1}{2} M_1\cdot M_2 \sin q_1 \sin q_2}$$
$$= \frac{Q_1 + Q_2 + \tfrac{1}{2} Q_1 * Q_2}{1 - \tfrac{1}{2} Q_1\cdot Q_2}. \tag{55}$$

The quantities Q_1, Q_2, Q are analogous to Gibbs' vector semi-tangent except that the whole angles q_1, q_2, q are used instead of the half angles.

We may call Q_1 the tangent vector of ϕ_1, Q_2 the tangent vector of Q_2 etc. Then (55) has the same general form as Gibbs' relation between the vector semi-tangents except for the $\frac{1}{2}$ and that the star product is used instead of the cross product. Similar results are obtained if M_1 and M_2 are anti-self-complementary.

If M_1 is self-complementary and M_2 anti-self-complementary

$$M_1 * M_2 = 0.$$

Hence if

$$\phi_1 = e^{q_1 I \cdot M_1}, \quad \phi = e^{q_2 I \cdot M_2}$$

ϕ_1 and ϕ_2 are commutative. That is *equiangular motions of opposite types are always commutative.* If, however, M_1 and M_2 are of the same type equation (54) shows that

$$\phi_1 \phi_2 = \phi_2 \phi_1$$

when and only when

$$\sin q_1 \sin q_2 \, M_1 * M_2 = 0.$$

If $\sin q_1 = 0$, $q_1 = 0$ or π and

$$\phi_1 = \pm I.$$

Hence *if neither dyadic is a multiple of the idemfactor, two equiangular rotations of the same type are commutative only when*

$$M_1 * M_2 = 0.$$

In this case M_1 and M_2 are expressible as multiples of the same completely perpendicular planes and so M_1 is a multiple of M_2.

15. **Expression of a rotation as a product of equiangular rotations.**

Any rotation in four dimensions can be expressed as the product of two equiangular rotations of opposite type. For, we have

$$\phi = e^{I \cdot M} = e^{I \cdot \frac{1}{2}(M + M_c) + I \cdot \frac{1}{2}(M - M_c)}.$$

Now the dyadics

$$I \cdot \tfrac{1}{2}(M + M_c), \qquad I \cdot \tfrac{1}{2}(M - M_c)$$

are commutative. Hence

$$\phi = e^{I \cdot \frac{1}{2}(M + M_c)} \cdot e^{I \cdot \frac{1}{2}(M - M_c)}$$

which is the resolution required. Conversely, this can be done, except for sign, in only one way. For suppose

$$\phi = \phi_1 \phi_2 = \phi_3 \phi_4,$$

ϕ_1 and ϕ_3 being equiangular rotations of one type, ϕ_2 and ϕ_4 of the opposite type. Then

$$\phi_3^{-1} \phi_1 = \phi_4 \phi_2^{-1}$$

Now $\phi_3^{-1} \phi_1$ and $\phi_4 \phi_2^{-1}$ are equiangular motions of opposite type. Since they are equal they must leave all self-complementary complexes invariant and also all anti-self-complementary ones. They then leave all complexes invariant and so

$$\phi_3^{-1} \phi_1 = \phi_4 \phi_2^{-1} = \pm I,$$

whence

$$\phi_1 = \pm \phi_3, \qquad \phi_2 = \pm \phi_4.$$

We shall call ϕ_1 and ϕ_2 the components of ϕ.

Suppose

$$\phi = \phi_1 \phi_2, \qquad \phi' = \phi_1' \phi_2',$$

ϕ_1 and ϕ_1' being equiangular rotations of one type and ϕ_2 and ϕ of the opposite type. Since ϕ_1 and ϕ_1' are comm utative with ϕ_2 and ϕ

$$\phi \phi' = \phi_1 \phi_1' \cdot \phi_2 \phi_2'.$$

The product of two rotations is thus obtained by multiplying together corresponding components. If ϕ and ϕ' are commutative, either

$$\phi_1 \phi_1' = \phi_1' \phi_1, \qquad \phi_2 \phi_2' = \phi_2' \phi_2$$

or

$$\phi_1 \phi_1' = -\phi_1' \phi_1, \qquad \phi_2 \phi_2' = -\phi_2' \phi_2.$$